Finite Difference Model for Vertical Axis Wind Turbines

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A numerical technique based on Patankar's "SIMPLER" algorithm is developed to determine the flow characteristics and performance of a two-dimensional vertical axis wind turbine. The conservation of mass and momentum equations are solved using a finite difference procedure without the necessity of introducing an irrotationality constraint. The computational domain is subdivided into control volumes in cylindrical coordinates and the turbine blades are modeled as a porous cylindrical shell of one control volume thickness. The characteristics of the turbine are computed and compared with previous investigations. The results show a very good agreement.

Nomenclature

a_p, a_E, a_W, a_N, a_S = coefficients in the discretized momentum equations, where a_p is the center grid coefficient and a_E, a_W, a_N, and a_S are the neighboring grid coefficients
B = number of turbine blades
b_r, b_θ = source terms in the respective discretized momentum equations
C = turbine blade chord
C_L, C_D = lift and drag coefficients, respectively
C_p = pressure coefficient on the cylinder
C_p = power coefficient of the turbine
CPU = central processor unit
f = aerodynamic forces summed in r and θ directions
H = turbine height ($H=1$ for two-dimensional rotors)
$\hat{i}, \hat{j}, \hat{k}$ = unit vectors in the Cartesian coordinate system X, Y, and Z, respectively
O = center of the turbine
p = static pressure
p' = pressure correction
R = radius of the turbine
r, θ, z = noninertial cylindrical coordinate system attached to the blade
$\hat{r}, \hat{\theta}, \hat{z}$ = unit vectors in the cylindrical coordinate system r, θ, and z, respectively
S_L, S_D = lift or drag forces felt by individual turbine control volumes
s_L, s_D = total lift or drag forces due to B blades
T_O = torque generated at the center of the turbine
UDMVAT = upwind difference model for vertical axis turbine
V_∞ = freestream wind velocity
v_{abs} = absolute velocity of the wind with respect to the inertial system fixed at the center of the turbine
v_{rel} = relative velocity of wind observed from the noninertial reference frame fixed to the blades
v_r, v_θ, v_z = components of velocity in r, θ, and z directions, respectively
v_θ' = relative tangential velocity of wind observed from the noninertial reference frame fixed to the blades
X, Y, Z = inertial reference frame attached to the center of the turbine O, where X is parallel to the freestream wind
ZLL = zero lift line
α = blade angle of attack with respect to the relative wind velocity v_{rel}
$\Delta\theta$, Δr = increment in θ and r directions (grid size in θ and r directions), respectively
λ = tipspeed ratio, $=\omega R/V_\infty$
ρ = density of air
σ = solidity of the turbine, $=BC/2R$
ϕ = general dependent variable
ω = rotational velocity of the turbine, rad

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Introduction

AFTER many years of inactivity, research in wind energy devices was given a large impetus with the onset of the oil crisis. Windmills are classified for the most part by the orientation of the axis of rotation, that is, either horizontal or vertical. The horizontal axis wind turbines are more common, an example being the propeller type. These machines must orient themselves to constantly face the wind vector in order to achieve good efficiencies. Recently, vertical axis turbines, although not classified as a new configuration because they were used in China hundreds of years ago, have been studied by various researchers using modern analysis techniques. Common examples of these vertical axis turbines are the Savonius and Darrieus turbines. In 1968, South and Rangi[1] reintroduced the Darrieus rotor concept, and many analytical models to predict the aerodynamic performance of this type of wind turbine have been formulated since. The present paper also considers vertical axis machines; however, the blades are straight in contrast to the "egg beater" blade shape typical of the Darrieus rotors.

The existing analytical models can be classified as either momentum or vortex models.

Momentum Models

Notable among the early momentum models are those proposed by Wilson and Lissaman,[2] Strickland,[3] Shankar,[4] and Templin.[5] In principle, these models replace the rotor by an equivalent actuator disk and equate the forces on the rotor blades to the change in streamwise momentum through the rotor. For lightly loaded rotors and within a certain tipspeed ratio range these models were adequate to determine the gross characteristics of the turbine. The single actuator disk models, however, tend to overestimate turbine performance for higher blade loadings.

In 1975, the simple momentum models were expanded to the tandem actuator concept introduced by Lapin,[6] and thus came the advent of upwind and downwind aerodynamic models. Important among these are the double multiple streamtube model of Paraschivoiu[7] and the cylindrical actuator concept of Loth and McCoy.[8] The actuators had different interference factors and all momentum models assumed that the flow through the rotor was straight and the flow velocities normal to the freestream direction were zero. Thus, none of the momentum models account for the diverging flow through the turbine.

Vortex Models

The vortex models are classified by their wake structure. They are either constrained wake models[9,10] or free vortex models.[11,12] Fixed wake models use a vortex-sheet concept that is locally independent of time. The forces on the blades are determined by the Kutta-Joukowski law. Although the method identifies the differences in the induced flow between the upwind and downwind blades, the constrained far-wake boundary condition limits it to parallel flow through the rotor and does not account for flow divergence.

The free vortex prediction models, initiated by Fanucci and Walters[11] and later investigated by Strickland et al.,[12] are the most accurate of all prediction models. This model represents the blade by a bound vortex sheet and the wake by force-free shed vortices and, hence, is called the free vortex model. The shed wake vorticity is allowed to convect downstream with the local stream velocity and therefore accounts for flow divergence. The uniqueness and chief advantage of the free vortex model stem from its ability to compute transient characteristics of the turbine. In addition, since the blade is represented by distributed bound vortices, fine details of the flow, such as blade-on-blade interaction, blade wake interaction, and the pressure distribution on the airfoil, can be determined. This leads to an accurate determination of the aerodynamic forces on the blade at the expense of large CPU run times.

The above discussions indicate that the momentum models predict the overall performance reasonably well within certain ranges of tipspeed ratios (tipspeed ratio λ, the nondimensional rotor tip velocity, is defined by the relation $\lambda = \omega R / V_\infty$) and blade solidity (solidity σ is defined as the ratio of the total area of the blades to that of the rotor and for a constant chord blade, $\sigma = BC/2R$), but do not illuminate the flow details in and around the turbine and do not account for flow divergence. The vortex models alleviated most of these problems but had to pay for the enormous complexity in terms of a great amount of computer time. Meaningful results of the free vortex models are also limited to low tipspeed ratios. Hence, a model accounting for flow divergence with less CPU run times than the vortex model would be beneficial in performing design studies economically. This need is fulfilled by the Upwind Difference Model for the Vertical Axis Turbine (UDMVAT) introduced herein.

Theoretical Formulation

The aerodynamic characteristics of the straight-bladed Darrieus rotor are complex. To facilitate the analysis of the basic problem, consider temporarily the simple case of a single-bladed vertical axis wind turbine as shown in Fig. 1.

The top view of the blade is shown at a radial distance R from the center O of the turbine. An inertial frame with $\hat{i}$, $\hat{j}$, and $\hat{k}$ as unit vectors is fixed at the center of the turbine to represent the Cartesian coordinates X, Y, and Z. The Z axis is aligned along the rotation axis of the turbine. A noninertial cylindrical polar coordinate system represented by unit vectors $\hat{r}$, $\hat{\theta}$, and $\hat{z}$ is attached to the blade.

The freestream wind $\vec{V}_\infty$ undergoes a gradual change to an absolute velocity $\vec{v}_{abs}$ as it approaches the blade. A steady-state condition is assumed in this analysis and the turbine is rotating with a constant angular velocity $\vec{\omega}$. Our main interest here is to determine the torque $\vec{T}_O$ about the origin O produced by the turbine. $\vec{T}_O$ is given by the relation $\vec{T}_O = \vec{R} \times \Sigma\vec{f}$, where $\Sigma\vec{f}$ is the vector sum of all forces acting on the blade. In order to determine $\Sigma\vec{f}$, it is necessary to find the relative velocity vector $\vec{v}_{rel}$. Thus the problem reduces to finding the vector velocity field in and around the turbine. $\vec{v}_{rel}$, as seen in the coordinate system attached to the blade, is a vector sum of two vectors, namely, $-\vec{\omega}R$, the velocity vector due to the turbine's rotation; and $\vec{v}_{abs}$, the absolute velocity of the wind with respect to the inertial coordinate system. Hence,

$$\vec{v}_{rel} = v_r\hat{r} + v'_\theta\hat{\theta} = (\vec{v}_{abs}\cdot\hat{r})\hat{r} + (-\omega R + \vec{v}_{abs}\cdot\hat{\theta})\hat{\theta}$$

In the above equation $-\omega R$ is a known quantity from the turbine's assumed rotational velocity and the only quantity to be determined is v_{abs}.

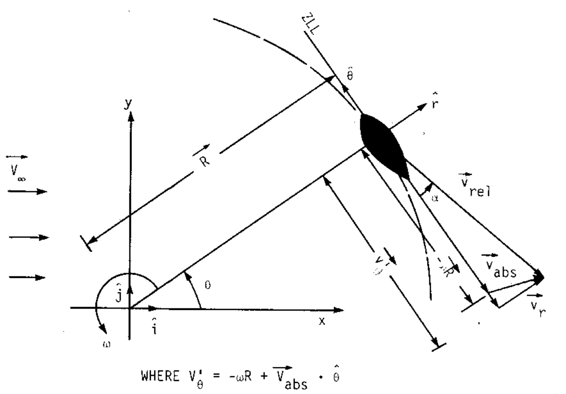

Fig. 1 Coordinate system. (ZLL = zero lift line.)

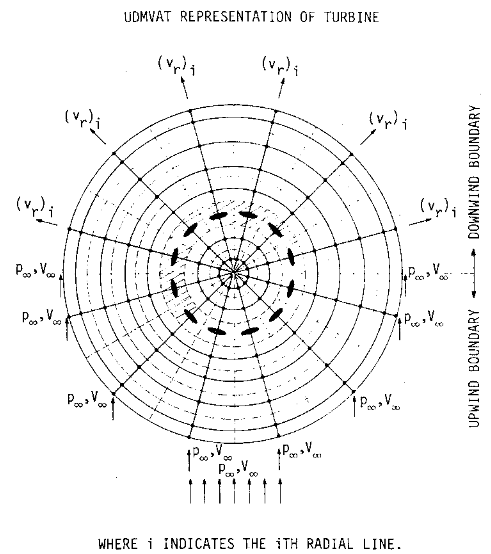

Fig. 2 UDMVAT representation of turbine.

UDMVAT—The Model

Since the problem is elliptic mathematically, a large but finite polar domain is assumed for computational purposes (see Fig. 2). The size of this domain (called the world size) will be discussed later.

Flow symmetry is not assumed and the entire circular domain of 2π rad is considered. This domain is subdivided into control volumes by a series of dotted lines orthogonal to the r and θ coordinate directions. The path of the blade develops a cylindrical shell. Since this analysis treats the flow as two-dimensional, only a slab of unit height is considered.

The control volumes that lie in the path of the turbine define the turbine boundary and are referred to as turbine control volumes. In reality, at any given time, the turbine blades occupy only specific turbine control volumes in its path. However, UDMVAT treats the effect of the blade in a time-averaged sense. Instead of considering the blade to be discrete with a certain solidity, it is modeled as a porous two-dimensional spinning cylindrical shell of equal solidity. At any given instant each turbine control volume contains a small blade and it is hypothesized that the cumulative effect of these small blades represents the finite number of turbine blades. In actuality the turbine blades are solid (nonporous) and offer a blockage to the flow, resulting in some loss of the impinging fluid's total head. The loss in total head can be thought of as "r" and "θ" momentum sinks. The strength of the sinks depends on the local relative velocity of the flow, the angular position of the control volume under consideration, the radius, the rotational velocity, and the solidity of the turbine. The UDMVAT formulation can be summarized with the help of Fig. 2 as follows:

1) A complete circular computational domain is chosen and subdivided into control volumes. The dashed lines represent the faces of the control volumes.

2) At the upwind boundary, freestream conditions are assumed.

3) At the downwind boundary, the flow must be fully expanded to freestream pressure; however, the outlet velocity is not known a priori.

4) At the turbine control volumes, special relationships must be developed to represent the effect of the blades on the flowfield.

Having stated and modeled the problem, the next task is to apply a numerical scheme for the solution of the flowfield. The flow is assumed to be inviscid and the analysis employs a finite difference scheme adopted from Patankar's "SIMPLER" algorithm,[13] which is based on an upwind differencing scheme. This algorithm is explained in detail in Ref. 13 and, therefore, discussion herein will be restricted to the highlights only.

Within the assumptions stated previously, the conservation equations written in cylindrical polar coordinates are:

Mass:

$$\frac{1}{r}\frac{\partial}{\partial r}(\rho v_r r)+\frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta)=0 \tag{1}$$

R momentum:

$$\frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r^2)+\frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta v_r)-\frac{\rho v_\theta^2}{r}=-\frac{\partial p}{\partial r} \tag{2}$$

θ momentum:

$$\frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r v_\theta)+\frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta^2)+\frac{\rho v_r v_\theta}{r}=-\frac{1}{r}\frac{\partial p}{\partial \theta} \tag{3}$$

The finite difference method used to solve the preceding equations is based on the control volume formulation. The solid lines in Fig. 2 are called grid lines, which connect the grid points. There is one control volume surrounding each grid point. The differential equations, Eqs. (1-3), are integrated over each control volume and the primitive variables p, v_r, and v_θ are conserved in every subdomain. For a velocity pressure coupled flow, a staggered grid system is known to give more realistic solutions and is adopted in the present study. Following the general procedure and notation used in Ref. 13, the conservation equations are discretized, and the resulting equations connecting the neighboring grid values to the center value have the form:

$$a_p\phi_p = a_E\phi_E + a_W\phi_W + a_N\phi_N + a_S\phi_S + b \tag{4}$$

where a is the coefficient connecting the variable ϕ, and b is the source term.

The discretized momentum conservation equations are substituted in the continuity equation to eliminate the velocity components and the result is an equation for pressure connecting the neighboring grid points. This pressure field denoted by p at all grid points must be corrected by a corrective pressure field p' to satisfy mass continuity. An approximate equation is formed for the p' field which resembles the p equation. Using the p' field, the pressure field is updated through the momentum equations subject to mass conservation constraint and, thus, the p' equation acts as a driver to the iterative scheme. This is the essence of the "SIMPLER" algorithm; the details of this solution procedure are not discussed here. It is sufficient here, to say that the discretized equations of p, p', v_r, and v_θ are Poisson equations resembling Eq. (4) and are solved using the line-by-line technique discussed in Ref. 13.

Evaluation of Turbine Sinks

The procedure discussed above is the general outline of the method for analyzing the fluid flow for all control volumes except for those containing the turbine. As mentioned previously, the blades in the ring of turbine control volumes behave as momentum sinks and, hence, a development for the treatment of these terms is in order. The source terms appearing in the r- and θ-momentum equations for any control volume, b_r and b_θ, respectively, have the units of force per unit length

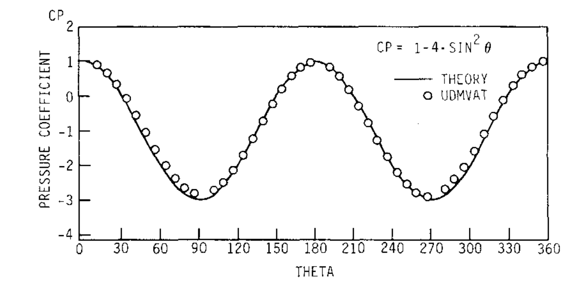

Fig. 3 Variation of pressure coefficient on a two-dimensional cylinder.

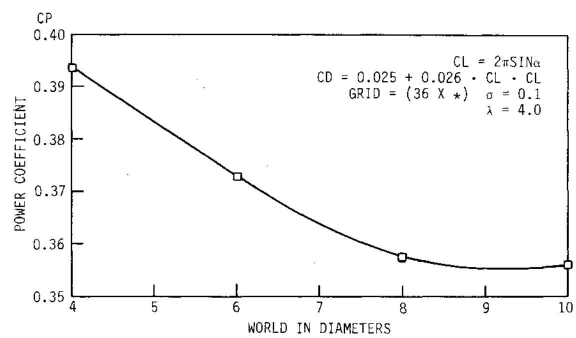

Fig. 4 Effect of world size on C_p.

given by the relation

$$b_r = -\rho v_\theta v_\theta \Delta r \Delta\theta, \qquad b_\theta = \rho v_r v_\theta \Delta r \Delta\theta \tag{5}$$

where Δr and $\Delta\theta$ are the grid sizes in the r and θ directions, respectively.

These source terms are valid for all control volumes save the turbine control volumes. For the case where the control volumes contain blades, the above expressions must be modified with additional source terms in order to account for the forces due to lift and drag created by the presence of the blades.

With regard to the lift, these additional source terms are given by

$$(s_L)_r = -(BC/2)\rho v_{\rm rel} v'_\theta C_L \tag{6}$$

$$(s_L)_\theta = (BC/2)\rho v_{\rm rel} v_r C_L \tag{7}$$

where B is the number of blades and C is the chord.

Averaging these forces over the circumference of the shell for one volume results in

$$(S_L)_r = \frac{(s_L)_r r\Delta\theta}{2\pi R} = -\frac{\sigma\rho C_L}{2\pi} v_{\rm rel} v'_\theta r\Delta\theta \tag{8}$$

$$(S_L)_\theta = \frac{(s_L)_\theta r\Delta\theta}{2\pi R} = \frac{\sigma\rho C_L}{2\pi} v_{\rm rel} v_r r\Delta\theta \tag{9}$$

where $(S_L)_r$ and $(S_L)_\theta$ are the additional source terms due to lift to be subtracted from b_r and b_θ, respectively, at the turbine control volumes. In a similar fashion, the source terms accounting for the drag are given by

$$(S_D)_r = \frac{\sigma\rho C_D}{2\pi} v_{\rm rel} v_r r\Delta\theta \tag{10}$$

$$(S_D)_\theta = \frac{\sigma\rho C_D}{2\pi} v_{\rm rel} v'_\theta r\Delta\theta \tag{11}$$

Thus, b_r and b_θ of the turbine control volumes are modified for every iteration.

The forces on the blades are given by

$$f_r = (S_L)_r + (S_D)_r, \qquad f_\theta = (S_L)_\theta + (S_D)_\theta$$

where

$$\vec{f} = f_r \hat{r} + f_\theta \hat{\theta}$$

and the torque is then given by

$$\vec{T}_O = \Sigma \vec{R} \times \vec{f}$$

where the summation is taken over all turbine control volumes. The power coefficient C_p of the turbine is computed using the relation

$$C_p = \frac{(\vec{T}_O \cdot \vec{\omega})}{\frac{1}{2}\rho V_\infty^3 (2RH)}$$

where H is the turbine height ($H = 1$ for two-dimensional rotors).

Results

To verify the computer code based on UDMVAT, the pressure coefficient for a two-dimensional cylinder in a uniform flow was computed using a fine-grid size and the result is compared with the potential flow solution in Fig. 3.

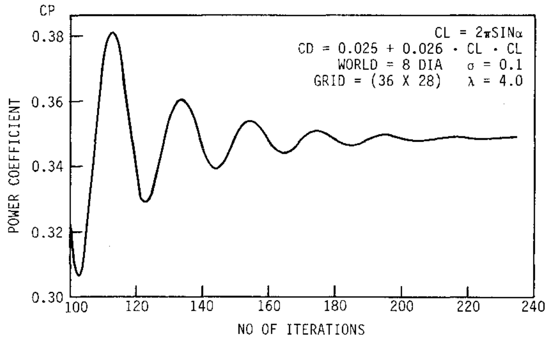

Fig. 5 Convergence of C_p with iterations.

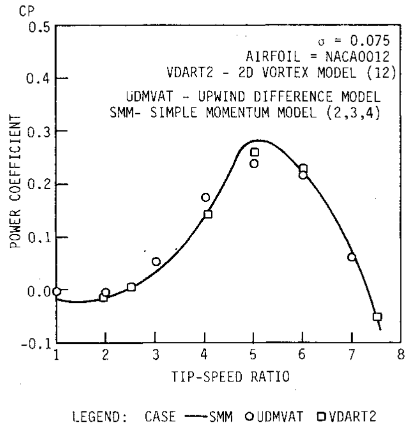

Fig. 6 Comparison of UDMVAT with vortex and momentum models.

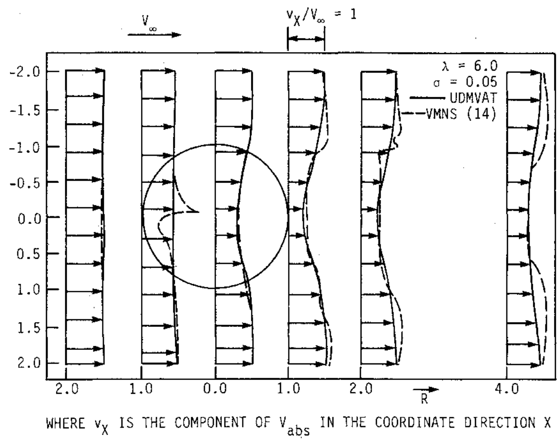

Fig. 7 Velocity profile near the turbine.

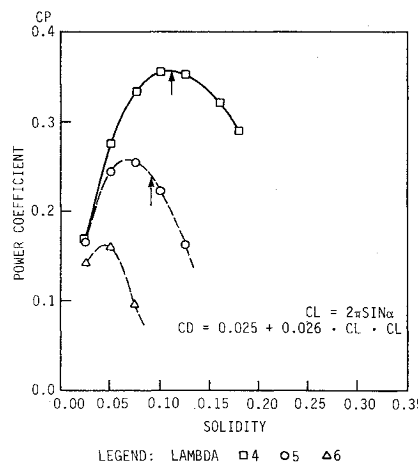

Fig. 8 Influence of solidity on performance predicted by UDMVAT.

These computations were made over a very small domain with boundary conditions precomputed from potential theory. The above exercise demonstrates that the computer code is successful in obtaining meaningful results when the outer boundary conditions are implemented correctly.

For the turbine problem it was necessary to first establish the size of the computational domain, which is referred to as the world size. The domain was considered to be large enough when the power coefficient C_p reached an asymptotic value. For all computations, the Δr grid size was chosen to be two-sevenths of the turbine radius. As shown in Fig. 4, C_p reached its asymptotic value at a world size of 8 turbine diameters. Since a uniform grid was employed, choosing the world size of 8 diameters fixed the number of r-direction grids to 28. A similar exercise was carried out for the θ-grid size and it was found that $\Delta\theta = 10$ deg was sufficient. For all other computations, the fineness of the grid was always maintained at 36×28 with a world size of 8 turbine diameters.

In Fig. 5, the C_p variation with number of iterations is presented and the convergence characteristics of C_p showed a remarkable similarity with the oscillations of a classical damped spring mass system. Figure 6 compares the turbine performance as predicted by UDMVAT with the vortex and simple momentum models. The comparison is very good. Velocity profiles at various stations were computed and the results showed good agreement with wake profiles predicted by VMNS.[14] In Fig. 7, the computer velocity profiles in the regions near the turbine are in close agreement with the free vortex method. Figure 8 shows the variation of C_p with different turbine solidities and is compared with Loth's theory.[15] Loth has developed a simple relationship that predicts the location for maximum C_p as a function of solidity and tipspeed ratio. This relationship is given by $\sigma\lambda = 0.45$ and is indicated in Fig. 8 by the vertical arrows. A typical computer run time to evaluate one C_p value on a AMDAHL V/7A is found to be 20 min. The details of this work are contained in Ref. 16.

Conclusions

A new technique to analyze the performance of a two-dimensional vertical axis wind turbine using a finite difference scheme has been developed. The method is unique in that the drag characteristics of the blades enter into the solution during the determination of the flowfield. In addition, the irrotationality assumption is not required. Computer run times are reasonably short and comparisons with other theories are good.

Acknowledgments

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